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LETTER TO THE EDITOR

Vortices and the prescribed curvature problem

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Abstract. The static Abelian Higgs model in an arbitrary two-dimensional geometry is considered. The consistency of a Bogomol'nyi bound to the energy functional is discussed and it is shown that the Bogomol'nyi equations are equivalent to a prescribed curvature problem.

The Landau-Ginzburg model [1] for a superconducing material which is uniform in one direction defines the usual two-dimensional Abelian Higgs model. It is known that, on the whole Euclidean plane, the energy functional for this model has a finite lower bound [2]. Moreover, for a certain choice of model parameters, which corresponds to a superconductor between type I and type II, this Bogomol'nyi bound can be saturated if a system of first-order equations are satisfied. These equations have been shown to be solved completely by vortex-like configurations and for these solutions the energy functional is stationary [3].

In this letter we consider the Abelian Higgs model for a superconducting shell of arbitrary shape. We show that, for the static problem, the energy functional, defined from the intrinsic geometry of the shell alone, a Bogomol'nyi bound is possible. However, the equations required to saturate this bound have only non-trivial solutions for a certain class of geometries (which essentially rules out the possibility of a bounded superconducting shell). For such geometry we show that the Bogomol'nyi equations are equivalent to a prescribed curvature problem for a two-dimensional Riemannian manifold.

The expression for the free energy of a superconductor occupying a region R of space as a functional of the complex valued order parameter $\phi=\phi_1+\mathrm{i}\phi_2=\phi(x^1, x^2, x^3)$ and the vector field representing the magnetic induction $\boldsymbol{A}=\boldsymbol{A}(x^1, x^2, x^3)$ is given as follows (in appropriate units of measurement)

$$\mathscr{E}(\phi, \boldsymbol{A})=\int_R [\tfrac{1}{2}|\boldsymbol{\nabla}\phi-\mathrm{i}\boldsymbol{A}\phi|^2+\tfrac{1}{2}(\nabla\times\boldsymbol{A})^2+\tfrac{1}{8}\mu(|\phi|^2-1)^2]\,\mathrm{d}_3\boldsymbol{x}. \tag{1}$$

The desciption of the state of the superconducting material is given by the stationary points of $\mathscr{E}$: $\delta\mathscr{E}=0$ and results in the following boundary value problem

$$-(\boldsymbol{\nabla}-\mathrm{i}\boldsymbol{A})^2\phi+\tfrac{1}{2}\mu(|\phi|^2-1)\phi=0, \tag{2}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})+\tfrac{1}{2}\mathrm{i}(\phi^*(\boldsymbol{\nabla}-\mathrm{i}\boldsymbol{A})\phi-\phi\{(\boldsymbol{\nabla}-\mathrm{i}\boldsymbol{A})\phi\}^*)=0, \tag{3}$$

$$\boldsymbol{n}\cdot(\boldsymbol{\nabla}-\mathrm{i}\boldsymbol{A})\phi|_{\partial R}=\boldsymbol{n}\times(\boldsymbol{\nabla}\times\boldsymbol{A})|_{\partial R}=0, \tag{4}$$

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where $\boldsymbol{n}$ denotes the unit normal vector to the surface bounding R, ∂R. We now suppose that the superconductor is a thin shell. Thus, in terms of some orthogonal curvilinear coordinate system on $\mathbb{R}^3$, (u^1, u^2, u^3), $u^i = u^i(x^1, x^2, x^3)$, the shell R may be written in the form: $0 < u^3 < \varepsilon d(u^1, u^2)$, $(u^1, u^2) \in M$. Here M denotes a region in the plane or the whole plane itself and ε, which represents the scale of the thickness of the shell, is assumed small in comparison with any other length scale associated with the shell. (We can accommodate some variability in the thickness of the shell by allowing $d(u^1, u^2)$ to be non-constant.) On the assumption that field variations transversally across the shell are negligible and using the boundary condition (4) on the upper and lower surfaces of the shell we have

$$\left(\frac{\partial}{\partial u^3} - \mathrm{i}\tilde{A}_3\right)\phi = \frac{\partial \tilde{A}_3}{\partial u^1} - \frac{\partial \tilde{A}_1}{\partial u^3} = \frac{\partial \tilde{A}_3}{\partial u^2} - \frac{\partial \tilde{A}_2}{\partial u^3} = 0, \tag{5}$$

throughout the shell. (Here $\tilde{A}_i = (\partial x^j/\partial u^i)A_j$.) We let the metric induced on the shell from the Euclidean metric on $\mathbb{R}^3$ be

$$\mathrm{d}s^2 = \sum_{\alpha,\beta=1}^{2} g_{\alpha\beta}(u^1, u^2)\, \mathrm{d}u^\alpha\, \mathrm{d}u^\beta,$$

where $(u^1, u^2) \in M$ is regarded as a local coordinate system on the shell and

$$g_{\alpha\beta}(u^1, u^2) = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}\bigg|_{u^3=0}.$$

Thus the energy functional (1) becomes

$$\begin{aligned} E(\phi, A) &= \mathscr{E}(\phi, A)/\varepsilon \\ &= \int_M \mathrm{d}u^1\, \mathrm{d}u^2 \sqrt{g}\, \rho \left\{ \tfrac{1}{2} g^{\alpha\beta} \left[\left(\frac{\partial}{\partial u^\alpha} - \mathrm{i}A_\alpha\right)\phi\right]^* \left[\left(\frac{\partial}{\partial u^\beta} - \mathrm{i}A_\beta\right)\phi\right] \right. \\ &\qquad \left. + \tfrac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} + \tfrac{1}{8}\mu(|\phi|^2 - 1)^2 \right\}. \end{aligned} \tag{6}$$

Here we have dropped the tilde on A, $F_{\alpha\beta} = \partial A_\beta/\partial u^\alpha - \partial A_\alpha/\partial u^\beta$ $\alpha, \beta = 1, 2$ and we have integrated over the u^3 variable, as the integrand is independent of u^3 to the lowest order in ε (the function $\rho(u^1, u^2)$ can be written in terms of the Jacobian of the transformation $(x^1, x^2, x^3) \to (u^1, u^2, u^3)$ and $d(u^1, u^2)$). The description of the superconducting shell is given by $\delta E = 0$ and results in the following boundary value problem

$$-\frac{1}{\sqrt{g}\,\rho}\left(\frac{\partial}{\partial u^\alpha} - \mathrm{i}A_\alpha\right)\left(g^{\alpha\beta}\sqrt{g}\,\rho\left(\frac{\partial}{\partial u^\beta} - \mathrm{i}A_\beta\right)\phi\right) + \tfrac{1}{2}\mu(|\phi|^2 - 1)\phi = 0, \tag{7}$$

$$\frac{1}{\sqrt{g}\,\rho}\frac{\partial}{\partial u^\sigma}(g^{\alpha\beta}g^{\gamma\sigma}\sqrt{g}\,\rho F_{\beta\gamma}) + \tfrac{1}{4}\mathrm{i}g^{\alpha\beta}\left\{\phi^*\left(\frac{\partial}{\partial u^\beta} - \mathrm{i}A_\beta\right)\phi - \phi\left[\left(\frac{\partial}{\partial u^\beta} - \mathrm{i}A_\beta\right)\phi\right]^*\right\} = 0, \tag{8a}$$

$$F_{\alpha\beta}|_{\partial M} = n^\alpha\left(\frac{\partial}{\partial u^\alpha} - \mathrm{i}A_\alpha\right)\phi|_{\partial M} = 0, \tag{8b}$$

where $\boldsymbol{n}$ is the unit normal to ∂M, the boundary of M.

On any two-dimensional manifold we may choose conformal coordinates. Doing this in our case the metric becomes $\mathrm{d}s^2 = \lambda^2(u^1, u^2)\, \mathrm{d}u^\alpha\, \mathrm{d}u^\alpha$ i.e. $g_{\alpha\beta} = \lambda^2\delta_{\alpha\beta}$, $\sqrt{g} = \lambda^2$, and λ^2 is nowhere zero in M. For the choice $\rho(u^1, u^2) = 1$ (which will hold henceforth) the energy functional (6) may be put in the form (using conformal coordinates)

$$E(\phi, A)=\int_M \mathrm{d}u^1\,\mathrm{d}u^2\,\lambda^2\bigg\{\tfrac{1}{2}\lambda^{-2}\bigg|\bigg[\bigg(\frac{\partial}{\partial u^1}-\mathrm{i}A_1\bigg)\pm\mathrm{i}\bigg(\frac{\partial}{\partial u^2}-\mathrm{i}A_2\bigg)\bigg]\phi\bigg|^2$$
$$+\tfrac{1}{2}\lambda^{-4}(F_{1,2}\pm\tfrac{1}{2}\lambda^2(|\phi|^2-1))^2+\tfrac{1}{8}(\mu-1)(|\phi|^2-1)^2\bigg\}\pm Q, \tag{9}$$

where Q is the boundary integral

$$Q=\frac{1}{2}\int_{\partial M}\bigg[A_\alpha-\mathrm{i}\phi^*\bigg(\frac{\partial}{\partial u^\alpha}-\mathrm{i}A_\alpha\bigg)\phi\bigg]\,\mathrm{d}u^\alpha. \tag{10}$$

Here ∂M denotes the boundary of M if M is a bounded region or the circle at infinity if M is $\mathbb{R}^2$. In the case $\mu=1$, which corresponds to a superconductor between type I and II, we can write the Bogomol'nyi bound: $E(\phi, A)\geqslant|Q|$ and we can saturate the bound when the first-order Bogomol'nyi system of equations is satisfied i.e.

$$[(\partial/\partial u^1-\mathrm{i}A_1)\pm\mathrm{i}(\partial/\partial u^2-\mathrm{i}A_2)]\phi=0, \tag{11}$$

$$F_{1,2}\pm\tfrac{1}{2}\lambda^2(|\phi|^2-1)=0. \tag{12}$$

The boundary conditions associated with these equations are

$$|\phi|^2|_{\partial M}-1=(\partial/\partial u^\alpha-\mathrm{i}A_\alpha)\phi|_{\partial M}=0. \tag{13}$$

Equation (11) can be solved for A_α

$$A_1=(\partial/\partial u^1)\,\mathrm{Arg}\,\phi\pm(\partial/\partial u^2)\ln|\phi|,$$
$$A_2=(\partial/\partial u^2)\,\mathrm{Arg}\,\phi\mp(\partial/\partial u^1)\ln|\phi|. \tag{14}$$

Then the boundary value problem defined by equations (11)-(13) reduces to one for the single undermined function $|\phi|^2$, namely

$$\Delta\ln|\phi|^2=|\phi|^2-1, \tag{15}$$

with, if M is bounded

$$|\phi|^2|_{\partial M}-1=0$$
$$(\partial/\partial u^1)|\phi|^2|_{\partial M}=(\partial/\partial u^2)|\phi|^2|_{\partial M}=0, \tag{16a}$$

and if $M=\mathbb{R}^2$

$$|\phi|^2\to 1, \qquad \text{as } |u|\to\infty. \tag{16b}$$

Here, Δ denotes the Laplacian on M: $\Delta\equiv\lambda^{-2}[(\partial/\partial u^1)^2+(\partial/\partial u^2)^2]$.

A condition which remain to be accounted for has to do with the topology of the fields. When ϕ, which is required to be at least twice differentiable on $\bar{M}=MU\partial M$, is restricted to the boundary it gives a continuous map $\partial M\to S^1$, the unit circle, if M is bounded, or from the circle at infinity to S^1 if $M=\mathbb{R}^2$. Letting the integer N be the winding number of this map we have from equations (10) and (16) that

$$Q=\tfrac{1}{2}\int_{\partial M}\frac{\partial\,\mathrm{Arg}\,\phi}{\partial u^\alpha}\,\mathrm{d}u^\alpha=\pi N, \tag{17}$$

and with the appropriate choice of $\pm$ in equations (9), (11) and (12) then $E=\pi|N|$.

There is a difficulty with the Bogomol'nyi system on M if M is bounded since the boundary conditions (16a) are overdetermined. Thus we will henceforth consider the case $M=\mathbb{R}^2$ and the boundary value problem for $|\phi|^2$ given by equations (15) and (16b). For simplicity we will also suppose that the metric on M is Euclidean asymptotically i.e. $\lambda^2\to 1$ as $|u|\to\infty$. The map $\phi|_{S^1_\infty}$ is given as a boundary condition and in solving equations (11) and (12) we are extending this map to obtain a continuous complex valued mapping ϕ on $\mathbb{C}=\mathbb{R}^2$. Thus if $N\neq 0$ we expect†, from equation (17), that Arg ϕ has $|N|$ isolated singularities (generically) in M and this is possible only if ϕ has $|N|$ zeros in M. In addition to equations (15) and (16) we have the conditions

$$|\phi|^2=0, \qquad \text{at } a_1, a_2, \ldots, a_n, \tag{18}$$

where $n=|N|$ and where we take $a_i\neq a_j$, $i\neq j$ (if two or more of the points a_1, $a_2,\ldots$ coincide, the multiplicity of the zero of $|\phi|^2$ must be counted).

For any metric $\gamma^2[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2]$ on a two-dimensional manifold the scalar curvature is given by

$$S(\gamma^2)=-\tfrac{1}{2}\gamma^2[(\partial/\partial u^1)^2+(\partial/\partial u^2)^2](\ln\gamma^2). \tag{19}$$

Let λ^2 be written as $1+[(\partial/\partial u^1)^2+(\partial/\partial u^2)^2]\beta$, where the function β is such that $-\infty<\beta<\infty$, $0<\lambda^2<\infty$ and $\lambda^2\to 1$, as $|u|\to\infty$ (in particular if $\lambda^2=1$ we take $\beta=0$). We then have

$$\lambda^2=[(\partial/\partial u^1)^2+(\partial/\partial u^2)^2]\ln(\exp(\tfrac{1}{4}|z|^2+\beta)), \tag{20}$$

where $z=u^1+\mathrm{i}u^2$, and equation (15) for $|\phi|^2$ can be written as the condition that the conformal deformation of the metric: $\exp(\tfrac{1}{4}|z|^2+\beta)[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2]$ by multiplying by $|\phi|^2$ preserve the scalar curvature i.e.

$$S(|\phi|^2\exp(\tfrac{1}{4}|z|^2+\beta))=S(\exp(\tfrac{1}{4}|z|^2+\beta)). \tag{21}$$

The metric $|\phi|^2\exp(\tfrac{1}{4}|z|^2+\beta)[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2]$ has singularities at the points a_1, $a_2,\ldots,a_n$ corresponding to the zeros of $|\phi|^2$. We can remove these singularities by defining the following metric

$$\gamma^2[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2]=\frac{|\phi|^2\exp(\tfrac{1}{4}|z|^2+\beta)[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2]}{|z-a_1|^2\ldots|z-a_n|^2}. \tag{22}$$

Equation (15) for $|\phi|^2$ can now be stated in the form of a prescribed curvature problem for the metric (22)

$$S(\gamma^2)=|z-a_1|^2\ldots|z-a_n|^2\exp(-\tfrac{1}{4}|z|^2-\beta)\lambda^2. \tag{23}$$

This prescribed curvature problem for γ^2 together with the asymptotic condition that

$$\gamma^2\to\exp(\tfrac{1}{4}|z|^2)|z|^{-2n}, \qquad \text{as } |z|\to\infty, \tag{24}$$

may be regarded as a special case of Dirichlet problem for the complex Monge-Ampere equation in a single complex variable z[4]. Theorems on existence and uniqueness for this problem have as yet only been proven for *bounded* domains [5]. Thus, unfortunately, we are unable to infer existence and uniqueness for the solution to equations (23) and (24). In fact the converse may be inferred if $\lambda^2=1$ ($\beta=0$). That is, the Bogomol'nyi problem (as given by equations (15), (16b), (17) and (18)), with $\lambda^2=1$

† If the solution for ϕ is real analytic as for the $\lambda^2=1$ case [3].

has already been shown to have a unique solution [3]. Therefore, the equivalent prescribed curvature (Dirichlet–Monge–Ampere) problem of equations (23) and (24) with $\lambda^2 = 1$ has also a unique solution on $\mathbb{R}^2$.

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